NAVAL No. S400.
R.A.F. Form 619

ROYAL AIR FORCE.

JAMES R. WILLIAMS. 3aR.
cot. Analysis

Notebook for use in Schools.

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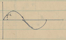
1947

Cct Analysis

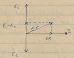
Bacuit Analysis.



$i = V_{max} \sin \omega t.$

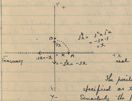
$I = I_{max} \sin \omega t.$

VECTOR METHOD.

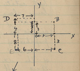


$$E_R^2 + (E_L - C_c)^2 = E_r^2.$$

j NOTATION



$j^4x = j^2 \times j^2 x$
$= -1 \times -1$
$= x$



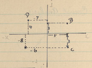
The position of the point B can be specified as X 5. y 3 or as $5 + j3$

Sumarlarly the point C can be specified as $5 + j3$. or more simply $5 - j3$

D $-7 + j4$

E $-6 - j8$

ARGAND DIAGRAM.



Let j be an operator such that when it operates on a vector OA it rotates 90° in the +ve direction so that it becomes the vector OB.

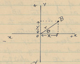
ie. $j \times OA = OB$

Taking j so defined, let it operate on the vector x, thus producing jx a vector of equal magnitude but rotated through 90° in the +ve direction, and being along the OY axis.

Let j again operate on the quantity jx to give is the quantity jx. Then from the defined properties of j. j^2x will be a vector of magnitude x. rotated 90° in the ~~position~~ direction from the position of the jx vector and therefore being on the O-X Axis. It is therefore indentical in magnitude & direction with a vector of $-x$

$\therefore j^2x = -x \quad j^2x = -x \quad j^2 = -1$

$\therefore j = \sqrt{-1}$



The position of B can be expressed as $x + jy$

The vector OB can be represented graphically as shown ie. in toto it can also be represented by it components in the two perpendicular directions OX, OY. ie. we can say that OB is the result of adding the quantities x & y vectorially. ie. $OB = x + y$ or $OB = x + jy$. Now the component of OB could also be written as OB. cos θ which is x. and OB sin θ which is y. ie. $OB = r\cos\theta + r\sin\theta$.

$= r\cos\theta + jr\sin\theta = r(\cos\theta + j\sin\theta)$

Real
Imaginary } Quantities.
Complex

Ways of representing complex quantities such as the vector OB..

① Graphically :- line drawn to represent it in Magnitude and direction

② Cartesian Form $x + jy$ ie series of real and imaginary components.

③ Polar Form. $r\angle\theta$ ie specifying magnitude on Modulus r and angle or arrangement θ which it makes with the assumed zero angle line.

Conversion from Cartesian form to Polar. We must find the Modulus r and the arrangement θ.

From the diagram $r = \sqrt{x^2 + y^2}$ NOT $\sqrt{x^2 + (jy)^2}$
which would be $\sqrt{x^2 - y^2}$

And $\theta = \tan^{-1} \frac{y}{x}$ NOT $\frac{jy}{x}$

N.B. $\tan^{-1}$ = a tangent whose angle is.

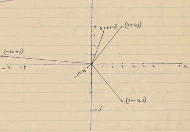
Eg i) Plot the following numbers on an Argand diagram and then put them in Polar Form.

$3 + 4j$, $3 - 4j$, $1{\cdot}2 + 3{\cdot}5j$, $-3{\cdot}2 + j$, $-{\cdot}5 - {\cdot}3j$

Eg ii) Change each of the following to the form $x + jy$

$3\angle 30^\circ$ $2{\cdot}3\angle 42$ $5{\cdot}2\angle -36^\circ$ $6{\cdot}3\,\overline{\angle 42^\circ}$

Eg I



Eg II. Change to the form $x + jy$

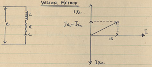
(i) $3\angle 30^\circ$ =

$jy = 3 \sin 30^\circ$
$x = 3 \cos 30^\circ$

$6{\cdot}3\,\overline{\angle 42^\circ}$

$-jy = 6{\cdot}3 \sin 42^\circ$
$x = 6{\cdot}3 \cos 42^\circ$

Vector Method



j Notation Method. Cartesian Form.

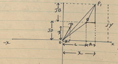
$$e = IR + jIX_L - jIX_C$$
$$e = I[R + j(X_L - X_C)]$$
$$\frac{e}{I} = Z = R + jX_L - jX_C$$
and $$\frac{e}{I} = Z = R + j(X_C - X_L)$$

Conversion to Polar Form

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \angle \tan^{-1} \frac{X_L - X_C}{R}$$

Addition of Complex Quantities :- Retangular Co-ordinate or j notation.

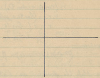
$$(A + jB) + (C + jD) = (A + C) + j(B + D)$$



$OP = (A + jB)$
$OQ = (C + jD)$
OP_1 = Resultant OP & OQ.
$OP_1 = OX + J \times Y$
$= X + jY$
$= C + A + j(D + B)$

Substraction of Complex Quantities

$$(A + Bj) - (C + jD) = (A - C) + j(B - D)$$



Multiplication :-

$$(A + jB)(C + jD) = AC + AjD + jBC + jB + jD$$
$$= AC + j(AD + BC) + j^2BD$$
$$= (AC - BD) + j(AD + BC)$$

$r_1 \angle \theta_1 \times r_2 \angle \theta_2 = r_1 r_2 \angle \theta_1 \theta_2$

Multiplication :- Expressed in Retangular coordinate of j notation form :-

$$r_1 \angle\theta_1 \times r_2 \angle\theta_2 = r_1 r_2 \angle\theta_1\theta_2$$

$$r_1(\cos\theta_1 + j\sin\theta_1) \times r_2(\cos\theta_2 + j\sin\theta_2)$$

$$r_1 r_2 = (\cos\theta_1 + j\sin\theta_1)(\cos\theta_2 + j\sin\theta_2)$$

$$r_1 r_2 = (\cos(\theta_1+\theta_2) + j\sin(\theta_1+\theta_2))$$

Division :-

$$(A + jB) \div (C + jD) = \frac{A+jB}{C+jD} = \frac{A}{C+jD} + \frac{jB}{C+jD}$$

To complete the division we Rationalise the denominator ie convert it from its complex form to a wholly real form. Note that to get the denominator into the desired form we can multiply it by anything we like provided we multiply the numerator & denominator by the same thing this does not alter the value of the fraction.

Note that $(a+b)(a-b) = a^2 - b^2$ and $a+b$ is called the congugate of $a-b$, and $a-b$ called the congugate of $a+b$.

Similarly $(a+jB)(a-jB) = a^2 - (jB)^2 = a^2 - j^2b^2 = a^2 - b^2 = a^2 + b^2$ ie Multiplying $a+jB$ by its congugate $a-jB$ gives the wholly real result a^2+b^2

$$\frac{A+jB}{C+jD} = \frac{A+jB}{C+jD} \times \frac{C-jD}{C-jD}$$

$$= \frac{(A+jB)(C-jD)}{C^2+D^2} = \frac{AC - jDA - jBC + BD}{C^2 D^2}$$

$$= \frac{AC + BD + j(BC - AD)}{C^2 D^2}$$

$$= \frac{AC+BD}{C^2+D^2} + j\frac{BC-AD}{C^2+D^2}$$

$$= x + jy.$$

Division Polar Notation

$$r\angle\theta_1 \div r_2\angle\theta_2 = \frac{r_1\theta_1}{R_2\theta_2} = \frac{r_1}{r_2}\angle\theta_1 - \theta_2$$

Proof :- $r_1\angle\theta = r_1\cos\theta_1 + jr_1\sin\theta = r_1(\cos\theta_1 + j\sin\theta_1)$

$$r_2\angle\theta_2 = r_2\cos\theta_2 + jr_2\sin\theta_2 = r_2(\cos\theta_2 + j\sin\theta_2)$$

$$\therefore \frac{r_1\theta_1}{r_2\theta_2} = \frac{[r_1(\cos\theta_1 + j\sin\theta_1)]}{[r_2(\cos\theta_2 + j\sin\theta_2)]} = \frac{r_1}{r_2}\,\frac{\cos\theta_1 + j\sin\theta_1}{\cos\theta_2 + j\sin\theta_2}.$$

$$\frac{r_1}{r_2} = \left[\frac{(\cos\theta_1 + j\sin\theta_1)(\cos\theta_2 - j\sin\theta_2)}{(\cos\theta_2 + j\sin\theta_2)(\cos\theta_2 - j\sin\theta_2)}\right]$$

$$= \frac{r_1}{r_2}\left(\frac{\cos\theta_1\cos\theta_2 - j\cos\theta_1\sin\theta_2 + j\sin\theta_1\sin\theta_2}{\cos^2\theta_2 + \sin^2\theta_2}\right)$$

$$= \frac{r_1}{r_2}\left(\frac{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + j(\sin\theta_1\cos_2 - \cos\theta_1\sin\theta_2)}{1}\right.$$

$$= \frac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + j\sin(\theta_1 - \theta_2)\right)$$

$$r_1\angle\theta \div r_2\angle\theta_1 = \frac{r_1\angle\theta_1}{r_2\angle\theta_2} = \frac{r_1}{r_2}\angle\theta_1 - \theta_2$$

Proof. Converting Result to Polar form.

$$\text{Modulas} = \sqrt{\frac{r_1^2}{r_2^2} \times [\cos^2(\theta_1-\theta_2) + \sin^2(\theta_1-\theta_2}} = \sqrt{\frac{r_1^2}{r_2^2}} \times 1 = \frac{r_1}{r_2}$$

Argument to $\tan^{-1} \frac{\sin(\theta_1 - \theta_2)}{\cos(\theta_1 - \theta_2)} = \tan^{-1} \tan \theta_1 \theta_2 = \theta_1 - \theta_2$

Polar Form is $\frac{r_1}{r_2} \angle \theta_1 - \theta_2$

sel *

$I = E\left(j\omega C + \frac{1}{\omega L j + R}\right) = E\left(j\omega C + \frac{R - j\omega L}{R^2 + \omega^2 L^2}\right)$

$I = E\left(\frac{j\omega C(R^2 + \omega^2 L^2) + R - j\omega L}{R^2 + \omega^2 L^2}\right)$

at Resonance $I = E(A + j0)$

$\therefore j\omega C(R^2 + \omega^2 L^2) - j\omega L = 0$

$= CR^2 + \omega^2 CL^2 = L$

$\omega^2 CL^2 = L - CR^2$

$\omega^2 = \frac{1}{LC} - \frac{R^2}{L^2}$

To find the Impedance at Resonance (DYNAMIC RESISTANCE)

$I_0 = E\left(\frac{R}{R^2 + \omega_0^2 L^2}\right) \quad \therefore Y_0 = \frac{I_0}{E} = \frac{R}{R^2 + \omega^2 L^2}$

$\therefore Z = \frac{1}{Y_0} = \frac{R^2 + \omega^2 L^2}{R}$

$= \frac{R^2 + \left(\frac{1}{LC} - \frac{R^2}{L^2}\right) L^2}{R} = R + \left(\frac{\frac{L}{C} - R^2}{R}\right)$

$Z_0 = \frac{L/C}{R} = \frac{L}{CR}.$

"Q" for a Component:- Defined as the ratio of series reactance to series Resistance ie. $\frac{X}{R}$

"Q" for a Circuit:- For any circuit "Q" is defined only at Resonance.

For a Series Cct. $Q = \frac{\text{Total Series Reactance of one Kind}}{\text{Total Series Resistance}}$

Both measured at Resonant frequency.

$Q = \frac{\omega_0 L}{R} = \frac{\frac{1}{\omega_0 C}}{R} = \frac{1}{\omega_0 CR}$

Hence $Q^2 = \frac{\omega_0 L}{R} \times \frac{1}{\omega_0 CR} = \frac{L}{CR^2}$

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

No single number can be partly real and partly imaginary

Hence $A + jB = 0$

$A = 0 \quad B = 0$

Note that if $A + jB = C + jD$ (Eq. i

then $A + jB - C - jD = 0$

$(AC) + j(B-D) = 0$

Applying above rule $AC = 0$ and $A = C$ (Eq. 1a

$(B-D) = 0$ and $B = D$ (Eq 1b

In any equation involving complex quantities the real part of one half of the equation will be equal to the half of the other equation. Similarly for the unreal part. Hence from the original complex equation two independent equations can be derived and that process is referred to as equating real and imaginary part.

From any bridge of the form:–

The condition of balance is $Z_1 Z_3 = Z_2 Z_4$

Hence for De Sauty bridge shown.

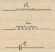
$$\left(R_x - \frac{j}{\omega C_x}\right) R = M\left(P - \frac{j}{\omega C_s}\right)$$

$$RR_x - \frac{jR}{\omega C} = MP - \frac{jM}{\omega C_s}$$

Equating Real and imaginary parts $RR_x = MP$ or $R_x = \frac{MP}{R}$

and $-\frac{jR}{\omega C_x} = -\frac{jM}{\omega C_s}$ or $\frac{R}{C_x} = \frac{M}{C_s}$ and $C_x = \frac{RC_s}{M}$

* Circuit Element — Impedance in Complex Notation

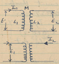


R

$+ j\omega L$

$\frac{-j}{\omega C}$ or $+\frac{1}{j\omega C}$ $\quad \frac{1}{j} = -j$

Mutual Inductance



$E_2 = -j\omega M I_1$

$E_1 = -j\omega M I_2$

If the Resistance of a cct. = R so that $I = \frac{E}{R}$ this property of the cct can equally well be expressed by saying the cct. has a Conductance G equal to $\frac{1}{R}$ so that $I = E \times G$. Similarly if we have a ~~sess~~ reactance X so that $I = \frac{E}{X}$, we can express its property by saying that it is a Susceptance. B equal to $\frac{1}{X}$ so that $I = E \times B$.

And if we have a cct of Impedance Z so that $I = \frac{E}{Z}$ we can express this property of of the cct by saying that it has an admittance Y equal to $\frac{1}{Z}$ so that $I = E \times Y$

In so far as R_X and Z are real and imaginary quantities, their Reciprocal would also be real and imaginary Quantities.

<u>Use of admittance - Conductance</u>

There are mainly of use in Parallel circuits.

(eg)

Y Circuit = G + B + Y

Proof:- Let E = Voltage applied to cct and I_G & I_Y. The current ~~therefore~~ through the branches G B & Y

Then $I_G = E \times G$. $I_B = E \times B$. $I_Y = E \times Y$. and

I total $= I_G + I_B + I_Y = E(G+B+Y)$

But Y cct = Admittance of whole cct

I total $= E \times Y$cct.

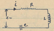
Hence $I_{Total} = E(G+B+Y) = E\,Y$cct

$\therefore G+B+Y = Y$ cct.

Y cct. = G + B + Y. NOTE $Z = \frac{1}{Y} = \frac{1}{G+B+Y}$

UNIT:- <u>MHOS</u> is reciprocal Ohms, a resistance of 10 ohms has a G of $\frac{1}{10}$ Mhos $\frac{1}{2}$ ohms = 2 Mhos

Reactance of 1000 apparent ohms has a Susceptance of $\frac{1}{1000}$ apparent mhos or 1 apparent Milli Mhos.



<u>Series Resonance.</u>

For cct shown $i = \frac{e}{Z} = \frac{e}{R + j\omega L - j \times \frac{1}{\omega C}}$

$$= \frac{e}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

Defining resonance as the condition which occurs when e & i are in Phase, we see that resonance ~~will~~ will occur when $\omega L = \frac{1}{\omega C}$. Because $i(R + j(\omega L - \frac{1}{\omega C})) = e$ and as e is a real quantity i must be wholly real to be in phase with i.

Putting $\omega L = \frac{1}{\omega C} = 0$ we have $\omega L = \frac{1}{\omega C}$ and $\omega^2 = \frac{1}{LC}$ and putting $(\omega L - \frac{1}{\omega C}) = 0$ in the original equation we have $i = \frac{e}{R}$ or $\frac{e}{i} = R$. that is impedance of cct at Resonance = R.

<u>Parallel Resonance</u>

$i = e(G + jB) = e\left[\frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)\right]$
*

For resonance e and i in Phase. the imaginary part of the R.H.S of the equation must be zero. That is $\omega C - \frac{1}{\omega L}$

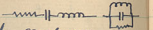
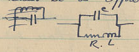
whence $\omega^2 = \frac{1}{LC}$ and $i = \frac{e}{R}$. So that $\frac{e}{i}$ = impedance of cct at Resonance = R.

* $i = e(G + jB) = e(G + B_{coil} + B_{cond}) = e\left(G + \frac{1}{j\omega L} + \frac{1}{\frac{-j}{\omega C}}\right)$

$= e\left(G - \frac{j}{\omega L} + \frac{\omega C}{-j}\right)$

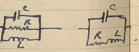
$= e\left(G \frac{-j}{\omega L} + \frac{j\omega C}{-j^2}\right) = e\left(G - \frac{J}{\omega C} + j\omega C\right) = e\left(G + j\left(\omega C - \frac{1}{\omega L}\right)\right)$

Problems

(1) Show that for these two circuit  The maximum energy stored by the condenser is equal to the maximum energy stored by the coil, when the cct is operating at the Resonant frequency. (b) Show that it is approximately true for this cct 

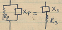
(2) Show that when this cct 

is at Resonance then is a circulating current I_c which is equal to Q times the feed current I_F.

(3) Show that for both these circuit 

The impedance of the whole cct at resonance = Q times impedance of one reactive arm.

(4) The two ccts. shown are equivalent. Find an exact expression for R_p in terms of X_s & R_s



X_p " " " X_s " R_s

R_s " " " X_p & R_p

X_s " " " X_p " R_p

Show that if $Q \gg 1$, $X_p = X_s$, $R_p = Q^2 R_s$

Answers.

(1) For the first two ccts at resonance

$$\omega^2 = \frac{1}{LC} \quad \therefore L = \frac{1}{\omega^2 C} \quad \therefore LI^2 = \frac{I^2}{\omega^2 C} = \frac{I^2}{\omega^2 C^2} \times C$$

$$\text{But } \frac{I}{\omega C} = I \times X_c = E_c \qquad \therefore LI^2 = \frac{I^2}{\omega^2 C^2} \times C = E_c^2 \times C$$

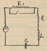
$$\text{So } \tfrac{1}{2} L I^2 = \tfrac{1}{2} C E_c^2$$

(2) 

$$I_F = \frac{E}{\frac{L}{CR}} = \frac{ECR}{L} \qquad Q = \frac{\omega_0 L}{R} \quad \therefore Q \times I_F = \frac{\omega_0 L}{R} \times \frac{ECR}{L}$$

$$= E\omega_0 C = I_c$$

Effect of damping on a series circuit



If without damping (S closed) $Q = Q_0 = \frac{\omega L}{R}$
Show that with extra damping
(S open) $Q = Q_1 = Q_0\left(\frac{R}{R+R_1}\right)$

Z with S closed $= R + R_1 + \left(j\omega L - \frac{1}{\omega C}\right)$
Q of the cct $= \frac{\omega L}{R+R_1}$

$Q_0 = \frac{\omega L}{R+R_1}$

Multiply $Q = \frac{\omega L}{R+R} \times R = \frac{\omega L R}{R(R+R_1)}$

$= Q_0\left(\frac{R}{R+R_1}\right)$

Parallel cct.

If with S open $Q = Q_0 = B_c R$
Show that $Q_1 = Q_0\left(\frac{R_1}{R_1+R}\right) = Q_0\left(\frac{R_1}{R_1+R}\right)$

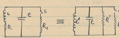
S open $Q = Q_0 = B_c R$ ——— ①
S closed $Q = Q_1 = B_c \times \frac{R R_1}{R+R_1}$ ——— ②

} From definition of Q for Parallel cct.

From ① $B_c = Q_0/R$
Substituting in ②
$Q_1 = \frac{Q_0}{R} \times \frac{R R_1}{R+R} = Q_0 \frac{R_1}{R+R_1}$

But Q for circuit is defined at resonance, and at resonance R_D the dynamic resistance of the cct $= R$

Hence. $Q_1 = Q_0\left(\frac{R_1}{R_D+R}\right)$



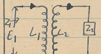
cct ① cct ②

For cct ① $R_D = \frac{L}{CR}$
Hence with S open $Q = Q_0$
" " S closed $Q = Q_1$
$Q_1 = Q_0 \frac{R_1}{\frac{L}{CR} + R_1}$

at Resonance $R_d = R'$ with S open
$\therefore R' = \frac{L}{CR}$

For cct ② $Q_1 = Q_0 \frac{R_1}{R_p + R_1}$ Just proved

$\therefore Q = Q_0 \frac{R_1}{\frac{L}{CR} + R_1}$



Problem to find Z_1 the impedance looking into L_1 (N.B. $Z_1 = \frac{E_1}{I_1}$)

$-E_1 + I_1 . j\omega L_1 - j\omega M I_2 = 0$
or $E_1 = I_1 . j\omega L_1 - I_2 j\omega M$ — Eq ①
$0 = I_2 (j\omega L_2 + Z_2) - j\omega M I$ — Eq ②
From Eq ② $I_2 = \frac{j\omega M I_1}{j\omega L_2 + Z_2}$

Substituting this value for I_2 in Eq ①

$$E_1 = I_1 j\omega L_1 - \frac{j\omega M I}{j\omega L_2 + Z_2} \times j\omega M$$

$$E_1 = I_1 \left(j\omega L_1 + \frac{\omega^2 M^2}{j\omega L_2 + Z_2} \right)$$

$$\frac{E_1}{I} = Z_1 = j\omega L_1 + \frac{\omega^2 M^2}{j\omega L_2 + Z_2}$$